MATHEMATICS 172
MATHEMATICAL MODELING FOR THE LIFE SCIENCES

BULLETIN INFORMATION

MATH 172 - Mathematical Modeling for the Life Sciences (3 credit hours)

Course Description:

Biological modeling with differential and difference equations; techniques of model modifications; analytic, numerical, and graphical solution methods; equilibria, stability, and long-term system behavior; geometric series; vectors, matrices, eigenvalues, and eigenvectors. Applications principally to population dynamics and compartment models.

Prerequisites: C or better in MATH 122 or MATH 141

SAMPLE COURSE OVERVIEW

Unless you have taken MATH 122 here at USC-Columbia you will possibly find that this course is very different from other math courses that you have taken. The course is not intended to be merely a collection of mathematical tools that you might someday, maybe, use in biology; rather it is an introduction to a way of thinking about biology that uses mathematics to create a conceptual and unified approach. Model building and model analysis, both quantitative and qualitative, will be as important as model solving. We form a mathematical model of a changing real world situation, such as population growth, use a variety of methods to analyze it, and then interpret our calculated results in the context of the original problem. We will analyze problems by using a blend of verbal, pictorial, numerical, graphical, and analytic methods(manipulation of formulas). Modeling is more comprehensive than problem solving; we will learn how models are built, as well as how to "read" them. One of the goals of the course is to enable you to look at scary formulas and to be able to dissect them into small and comprehensible pieces, so that even if you don't "do the math" yourself, you will be able to follow what has been done. Finally, in the real world, solutions must be communicated effectively, both in writing and orally, and you will get some practice doing this.

We begin with an introduction to the idea of modeling a system (why do it?), how to build models starting with really simple ones and gradually building up to more complex ones, and how to interpret models that you can pull off the shelf. Models come in two flavors: discrete and continuous; we will explore how these are similar, how they are different, and why one might choose one over the other. The discrete ones are usually called difference equations; the continuous ones are called differential equations (DE's) and these you have already been solving in your calculus classes! Every time that you concluded something about a function, say P (t) population as a function of time, from information about its rate of change P (t) you were actually solving a DE.

One of our main tools will be the study of the equilibria (stationary values) of a model, what these are, how to compute them, how to determine if they are stable or not, and how they are related to the long term behavior of a system. Put crudely, we want to use models to make predictions such as whether a population (or an epidemic) will boom (turn into a pandemic), go extinct (die out), or fluctuate around a certain level.

Next we will learn a little bit about matrices and vectors, emphasizing the intuitive geometric point of view, and then go on to selected topics involving multi-species or metapopulation models that use these basic tools. We will be very selective in what we cover; although the book is small, it is packed and we won't cover more than the first six chapters, perhaps with chapter 8 in place of chapter 4. Notice that the organization is by biological topic, not mathematical. All along we will work with tables of data, or verbal descriptions of problems, to build our models, and we will use our calculators to make educated guesses about the qualitative behavior of the solutions. You will be expected to gradually recognize for yourself when the use of technology is appropriate, and when hand computation and exact algebra or calculus is necessary.

ITEMIZED LEARNING OUTCOMES

Upon successful completion of Math 172, students will be able to:

1. Demonstrate understanding of the following concepts:
    a. Differential and difference equations
    b. Techniques of model modifications
    c. Analytic, numerical, and graphical solution methods
    d. Equilibria, stability, and long-term system behavior
    e. Geometric series
    f. Vectors, matrices, eigenvalues, and eigenvectors, with applications principally to population dynamics and compartment models.
2. Apply these concepts to solve problems drawn from biological modeling.

SAMPLE REQUIRED TEXTS/SUGGESTED READINGS/MATERIALS

1. A Primer of Ecology by Nicholas Gotelli, Sinauer Press, 2007, 4th ed., 2008.
    a. As you can tell from the title this text will support BIOL 301, if taken concurrently, and will be a great warm-up if taken before 301. That said, I will try to work in some models from other parts of biology, although, truth to tell, it is a lot easier to understand population counts than concentrations or moles (the chemical kind), etc. This text contains some genuine biology, but you don't need any specialized knowledge of ecology to use it. You will also need a graphing calculator (TI-83 preferred) that can handle sequences (discrete models, recurrence relations, discrete dynamical systems, these all mean much the same thing).

SAMPLE ASSIGNMENTS AND/OR EXAM

1. **Three Tests**

   a. Three major tests will be given. The final exam is optional. It will be divided into three parts corresponding to the three exams, and you may do as many of these parts as you like.

2. **Quizzes**
   a. At least seven eight-point quizzes will be given; the six highest scores will be counted. No make-ups will be given for quizzes or exams, but each of the three segments on the final exam (scaled up to be out of 100) will be used to replace the score for the corresponding in-class exam, provided this helps you.

3. **Two Projects**
   a. There will be two group projects.
4. **Homework**

SAMPLE COURSE OUTLINE WITH TIMELINE OF TOPICS, READINGS/ASSIGNMENTS, EXAMS/PROJECTS

Class 1: Review of proportions and units. Scaling. Argument that weight scales as the cube of the length, area as the square of length etc.
Reading: Instructor's notes.

Class 2: More on proportion and units. Understanding that if for a function y = f(x) doubling x doubles y, tripling x, triples y, etc., then y is proportional to x. If when x is doubled, y is multiplied by 22, if x is tripled then y is multiplied by 32 etc. Then y is proportional to x2 . Multivariate extensions of these ideas.
Reading: Instructor's notes.

Class 3: Using proportion and scaling to predict weights and other attributes. (For example if a 12 inch trout weighs 1 pound, then predict the weight of an 18 inch trout.)
Reading: Instructor's notes.

Class 4: More on scaling. Galileo's argument that the crushing pressure of material places a limit on its size (i.e. the height of a tree is limited by how much pressure its wood can withstand).
Reading: Instructor's notes.

Class 5: Basics of continuous exponential growth. Determination of an exponential by two data points. The rate equation for exponentials.
Reading: Gotelli, Chapter 1

Class 6: The intrinsic growth as the constant of proportionality between population size and per capita growth rate. Examples of the rate of exponential growth and decay
Reading: Gotelli, Chapter 1

Class 7: The "afine model" $dN/dt = rN - S$. Interpretation of S as a "stocking rate". Discussion of variants on this model. Determining the equilibrium points by looking for constant solution.
Reading: Instructor's notes.

Class 8: Discrete exponential growth. The difference equation defining discrete exponential growth. Discussion of when to use discrete versus continuous exponential models.
Reading: Gotelli, Chapter 1

Class 9: Affine models for discrete exponential growth. Equilibrium solutions as the constant solutions.
Reading: Gotelli, Chapter 1

Class 10: Autonomous first order differential equations, $dN\ dt = f(N)$. Equilibrium solutions both as the constant solutions and as the roots of $f(N) = 0$. Lots of examples.
Reading: Instructor's Notes.

Class 11: More on autonomous first order equations. Graphing solutions from understanding the sign changes of $f(N)$ in $dN/dt = f(N)$. Using this information to determine if an equilibrium point is stable or unstable. Lots of examples.
Reading: Instructor's notes.

Class 12: Continuous logistic growth. Deriving the model and analysis of the equilibrium points. Variants on the model such as harvesting from a logistic population.
Reading: Gotelli, Chapter 2

Class 13: Review

Class 14: Test 1

Class 15: Discrete dynamical systems $N_{t+1} = f(N_t)$. Equilibrium solutions both as constant solutions and as roots of $f(N) = N$. Pictorial analysis by cobweb diagrams.
Reading: Instructor's notes.

Class 16: Analysis of equilibrium points to a discrete dynamical system being stable (when $|f'(N)| < 1$) or unstable (when $|f'(N)| > 1$). How to determine this on the calculator.
Reading: Instructor's notes.

Class 17: Examples of discrete dynamical systems with no stable equilibrium points. Discussion of periodic and chaotic solutions, using as the main example the discrete logistic equation. Discussion of why discrete models can have much more complicated behavior than continuous models.

Reading: Instructor's notes, Gotelli, Chapter 2

Class 18: Age structured population growth. Starting with loop diagrams show that given an age distribution one year the age distribution for the next year can be computed.
Reading: Instructor's notes, Gotelli, Chapter 3.

Class 19: Showing that the information in a loop diagram can be coded into a matrix and that multiplication of the vector of the age distribution in one year by this matrix gives the age distribution for the next year. The matrix equation $n(t + 1) = An(t)$.
Reading: Instructor's notes, Gotelli, Chapter 3

Class 20: The basics of matrix algebra on the calculator. Powers of matrices. Examples showing that with a non-negative matrix and positive vector that the "distribution" stabilizes.
Reading: Instructor's notes, Gotelli, Chapter 3

Class 21: The per capita growth rate, r, as the number so that $n(t+ 1) \approx (1+r)n(t)$ for large t. Optional: Relating the stable distribution and pre capita to the positive eigenvalue and eigenvector.
Reading: Gotelli, Chapter Instructor's notes. Gotelli, Chapter 3.

Class 22: Derivation of the Euler-Lotka equation and examples.
Reading: Gotelli, Chapter 3.

Class 23: Variations the age structured model such as stage structured models.
Reading: Gotelli, Chapter 3

Class 24: More on stage structured populations. Showing that the even though the Euler-Lotka equation no longer holds, the per capita growth rate can still be found by looking at large powers of the matrix involved.
Reading: Instructor's notes.

Class 25: Metapopulations. A bit of probability, and discussion of examples. Definition of the extinction probability, pe, the probability, of colonization, pi, and the fraction, f(t) of sites that are populated in year t.
Reading: Gotelli, Chapter 3

Class 26: Derivation of the equation $df/dt = pi(1 - f) - pef$ for the island-mainland and related models. Finding the stable equilibrium points.
Reading: Gotelli, Chapter 4

Class 27: Discussion of metapopulations under the assumption of internal colonization. Derivation of $df/dt = if(1 - f) - pef$ for this model. Why extinction can occur in this model, but not the island-mainland model.
Reading: Gotelli, Chapter 4

Class 28: Review

Class 29: Test 2

Class 30: Two dimensional systems of autonomous differential equations using the Lotka-Volterra equations $dx/dt = r1x\ (K1-x-\alpha y)/K1$, $dy/dt = r2y\ (K2-\beta x-y)/K2$ for competing species as the main example.
Reading: Instructor's notes, Gotelli, Chapter 5.

Class 31: More on two dimensional systems. Equilibrium points and the notion of stable and unstable points. How in the case of the Lotka-Volterra equations for competing species the analysis of stability can be done in an elementary manner from the phase diagram.
Reading: Instructor's notes. Gotelli, Chapter 5.

Class 32: Detailed analysis of the Lotka-Volterra equations for competing species and a discussion of coexistence and completive exclusion.
Reading: Gotelli, Chapter 5.

Class 33: Variants on the Lotka-Volterra equations for competition and the phase plane determines stability in these more complicated models.
Reading: Instructor notes, Gotelli, Chapter 5.

Class 34: Predation. The Lotka-Volterra equations for a predator-prey system. Analysis of the system by the phase diagram, location of the equilibrium points. The periodicity of solutions.
Reading: Instructor's notes, Gotelli, Chapter 6

Class 35: More on the Lotka-Volterra model for predator-prey. The time series for the solutions showing that the predator and prey graphs differ by a quarter of a cycle.
Reading: Gotelli, Chapter 6.

Class 36: Variants on the Lotka-Volterra model for predator-prey. In particular the model where the pray grows logistic in the absence of the predator. In this case there are three equilibrium points and the one where both species are present has a stable equilibrium point, rather than periodic solutions.
Reading: Gotelli, Chapter 6.

Class 37: Review

Class 38: Test 3
39–42 Topics of the instructor's choice.

Final Exam Final exam according to University exam schedule